

CONSTRUCTION OF μ -NORMAL SEQUENCES

MANFRED G. MADRITSCH AND BILL MANCE

ABSTRACT. In the present paper we want to extend Champernowne's construction of normal numbers to provide normal numbers for different numeration systems where restrictions are imposed on the digital expansion. We present a construction together with estimates and examples for normal numbers with respect to Lüroth series, continued fractions expansion or β -expansion.

1. INTRODUCTION

Let $q \geq 2$ be a positive integer, then every real $x \in [0, 1]$ has a q -adic representation of the form

$$x = \sum_{h=1}^{\infty} d_h(x) q^{-h}$$

with $d_h(x) \in \mathcal{D} := \{0, 1, \dots, q-1\}$. We call a number $x \in [0, 1]$ normal with respect to the base q if for any $k \geq 1$ and any block $B = b_1 \dots b_k$ of k digits the frequency of occurrences of this block tends to q^{-k} . In particular, let $N_n(B, x)$ be the number of occurrences of B among the first n digits, *i.e.*

$$N_n(B, x) = \# \{0 \leq h < n : d_{h+1}(x) = b_1, \dots, d_{h+k}(x) = b_k\}.$$

Then we call $x \in [0, 1]$ normal of order k in base q if for every block B of length k we have

$$\lim_{n \rightarrow \infty} \frac{N_n(B, x)}{n} = q^{-k}.$$

Furthermore we call a number normal if it is normal of order k for every $k \geq 1$ and absolutely normal if it is normal in every base $q \geq 2$.

In 1909 Borel [4] showed that almost every real is absolutely normal. This motivated people to look for a concrete example of a normal number. It took more than 20 years until 1933 when Champernowne [5] provided the first explicit construction by showing that the number

0.12345678910111213141516

is normal to base 10. This construction was generalized to different numeration systems such as the Gaussian integers by Dumont *et al.* [7] or the continued fractions expansion by Adler *et al.* [1]. Another generalization to β -expansion is due to Ito and Shiokawa [8]. However, since in these numeration systems not every sequence of digits is allowed, their construction does not provide an admissible number.

The aim of the present paper is on the one hand to apply ideas of Altomare and Mance [2] and Mance [9, 10] in full generality. This will provide us on the one hand with a generalization of the construction of Champernowne for obtaining μ -normal numbers and on the other hand we want to construct admissible normal numbers for β - and similar expansions having digital restrictions.

We want to emphasize that we only present some examples of applications of this generalization. Since our examples deal with several issues in the construction like infinite digit set, restriction on the representations and irrational probability of digits, it should be possible for the reader to construct normal numbers for any positional number system.

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2. DEFINITIONS AND STATEMENT OF RESULTS

A *block of length k in base b* is an ordered k -tuple of integers in $\{0, 1, \dots, b-1\}$. A *block of length k* will be understood to be a block of length k in some base b . A *block* will mean a block of length k in base b for some integers k and b . Given a block B , $|B|$ will represent the length of B . Given blocks B_1, B_2, \dots, B_n and integers l_1, l_2, \dots, l_n , the block $B = l_1 B_1 l_2 B_2 \dots l_n B_n$ will be the block of length $l_1 |B_1| + \dots + l_n |B_n|$ formed by concatenating l_1 copies of B_1 , l_2 copies of B_2 , all the way up to l_n copies of B_n . For example, if $B_1 = (2, 3, 5)$ and $B_2 = (0, 8)$ then $2B_1 1B_2 = (2, 3, 5, 2, 3, 5, 0, 8)$.

Definition 2.1.¹ A *weighting* μ is a collection of functions $\mu^{(1)}, \mu^{(2)}, \mu^{(3)}, \dots$ with $\sum_{j=0}^{\infty} \mu^{(1)}(j) = 1$ such that for all k , $\mu^{(k)} : \{0, 1, 2, \dots\}^k \rightarrow [0, 1]$ and

$$\mu^{(k)}(b_1, b_2, \dots, b_k) = \sum_{j=0}^{\infty} \mu^{(k+1)}(b_1, b_2, \dots, b_k, j) = \sum_{j=0}^{\infty} \mu^{(k+1)}(j, b_1, b_2, \dots, b_k).$$

For blockes $B = b_1 \dots b_k$, we will write $\mu(B)$ in place of $\mu^{(k)}(b_1, b_2, \dots, b_k)$.

A block B is μ -*admissible* if $\mu(B) \neq 0$. Let \mathcal{D}_μ denote the set of μ -admissible blocks and let $\mathcal{D}_{\mu,k}$ denote the set of μ -admissible blocks of length k . Given blocks B and Y we will let $N_n(B, Y)$ denote the number of times a block B occurs starting in position no greater than n in the block Y . We will often write $N(B, Y)$ in place of $N_{|Y|}(B, Y)$. A sequence of weightings $(\mu_i)_{i=1}^{\infty}$ *converges* to μ (written $\mu_i \rightarrow \mu$) if $\mathcal{D}_{\mu_i} \subset \mathcal{D}_\mu$,² $\mu_i(B)$ is eventually non-increasing, and $\mu_i(B) \rightarrow \mu(B)$ for all blocks B .

Definition 2.2. Suppose that $0 < \epsilon < 1$, k is a positive integer and μ is a weighting. A block of digits Y is (ϵ, k, μ) -*normal*³ if for all $t \leq k$ and blocks B in $\mathcal{D}_{\mu,t}$, we have

$$(2.1) \quad \mu(B)|Y|(1 - \epsilon) \leq N(B, Y) \leq \mu(B)|Y|(1 + \epsilon).$$

For convenience, we define the notion of a block friendly family:

Definition 2.3. A *block friendly family* is a sequence of 4-tuples $W = ((l_i, \epsilon_i, k_i, \nu_i))_{i=1}^{\infty}$ with non-decreasing sequences of non-negative integers $(l_i)_{i=1}^{\infty}$ and $(k_i)_{i=1}^{\infty}$ such that $(\nu_i)_{i=1}^{\infty}$ is a sequence of weightings and $(\epsilon_i)_{i=1}^{\infty}$ strictly decreases to 0.

Definition 2.4. Let $W = ((l_i, \epsilon_i, k_i, \mu_i))_{i=1}^{\infty}$ be a block friendly family and let μ be a weighting. If $\lim k_i = K < \infty$, then let $R(W) = \{0, 1, 2, \dots, K\}$. Otherwise, let $R(W) = \{0, 1, 2, \dots\}$. If $(X_i)_{i=1}^{\infty}$ is a sequence of blocks such that $|X_i|$ is non-decreasing and X_i is (ϵ_i, k_i, ν_i) -normal, then $(X_i)_{i=1}^{\infty}$ is said to be (W, μ) -*good* if $\nu_i \rightarrow \mu$ and for all k in R the following hold:

$$(2.2) \quad \frac{1}{\epsilon_{i-1} - \epsilon_i} = o(|X_i|);$$

$$(2.3) \quad \frac{l_{i-1}}{l_i} \cdot \frac{|X_{i-1}|}{|X_i|} = o(i^{-1});$$

$$(2.4) \quad \frac{1}{l_i} \cdot \frac{|X_{i+1}|}{|X_i|} = o(1).$$

Before stating our main theorem we need the definition of μ -normal numbers.

Definition 2.5. Let μ be a weighting and k be a positive integer. Then we call an infinite sequence $X = (X_i)_{i=1}^{\infty}$ μ -normal of order k if for every block B of length k we have

$$\lim_{n \rightarrow \infty} \frac{N_n(B, X)}{n} = \mu(B).$$

¹Postnikov [12] discusses normality in base 2 with respect to different weightings.

²A version of our main theorem is still true if we drop the condition $\mathcal{D}_{\mu_i} \subset \mathcal{D}_\mu$, but every example we will consider has this property.

³Definition 2.2 is a generalization of the concept of (ϵ, k) -normality, originally due to Besicovitch [3].

Similar to above we call a sequence μ -normal if it is μ -normal of order k for every $k \geq 1$.

Now we are able to state our main theorem.

Main Theorem 2.1. Let W be a block friendly family and $(X_i)_{i=1}^\infty$ a (W, μ) -good sequence. If $k \in R(W)$ then X is μ -normal of order k . If $k_i \rightarrow \infty$, then $X = l_1 X_1 l_2 X_2 \dots$ is μ -normal.

The proof of this theorem will proceed in several steps. In the following section we build our toolbox where we present and prove all the lemmas we need in the proof of Main Theorem 2.1. In Section 4 we finally prove Main Theorem 2.1. Our construction is explained in Section 5. Since for different numeration systems we need different block friendly families we provide several examples for applications of our construction in Section 6.

3. TECHNICAL LEMMAS

We will proceed in a similar manner to Mance [9]. For this section, we will fix a block friendly family W and a (W, μ) -good sequence (X_i) . Put

$$L_i = \sum_{j=1}^i l_j |X_j|$$

For a given n , the letter $i = i(n)$ will always be understood to be the positive integer that satisfies $L_i < n \leq L_{i+1}$. This usage of i will be made frequently and without comment. Let $m = n - L_i$, which allows m to be written in the form

$$m = \alpha |X_{i+1}| + \beta$$

where α and β are integers satisfying

$$0 \leq \alpha \leq l_{i+1} \text{ and } 0 \leq \beta < |X_{i+1}|.$$

Thus, we can write the first n digits of X in the form

$$l_1 X_1 l_2 X_2 \dots l_{i-1} X_{i-1} l_i X_i \alpha X_{i+1} 1 Y,$$

where Y is the block formed from the first β digits of X_{i+1} .

For a block B , let

$$\phi_n(B) = \sum_{j=1}^i l_j |X_j| \nu_j(B) + m \nu_{i+1}(B).$$

As $\lim_{n \rightarrow \infty} \frac{\phi_n(B)}{n} = \mu(B)$, X is μ -normal if and only if

$$\lim_{n \rightarrow \infty} \frac{N_n(B, X)}{\phi_n(B)} = 1$$

for all blocks B .

Given a block B of length k in $R(W)$, we will first get upper and lower bounds on $N_n(B, X)$, which will hold for all n large enough that $k \leq k_i$. This will allow us to bound

$$(3.1) \quad \left| \frac{N_n(B, X)}{\phi_n(B)} - 1 \right|$$

and show that

$$\lim_{n \rightarrow \infty} \frac{N_n(B, X)}{\phi_n(B)} = 1.$$

We will arrive at upper and lower bounds for $N_n(B, X)$ by breaking the first n digits of X into three parts: the initial block $l_1 X_1 l_2 X_2 \dots l_{i-1} X_{i-1}$, the middle block $l_i X_i$ and the last block $\alpha X_{i+1} 1 Y$.

Put

$$\kappa = \kappa(n) = (L_{i-1} + k(l_i + 1) + (1 + \epsilon_i) \nu_i(B) l_i |X_i|) + ((1 + \epsilon_{i+1}) \nu_{i+1}(B) |X_{i+1}| + k) \alpha + \beta.$$

Lemma 3.1. If $k \leq k_i$ and $B \in \mathcal{D}_{\nu_i, k}$, then

$$(1 - \epsilon_i)\nu_i(B)l_i|X_i| + (1 - \epsilon_{i+1})\nu_{i+1}(B)\alpha|X_{i+1}| \leq N_n(B, X) \leq \kappa.$$

Proof. We can estimate $N_m(B, l_{i+1}X_{i+1})$ by using the fact that $k \leq k_{i+1}$ and X_{i+1} is $(\epsilon_{i+1}, k_{i+1}, \nu_{i+1})$ -normal so that $(1 - \epsilon_{i+1})\nu_{i+1}(B)|X_{i+1}| \leq N(B, X_{i+1}) \leq (1 + \epsilon_{i+1})\nu_{i+1}(B)|X_{i+1}|$. An upper bound for $N_m(B, l_{i+1}X_{i+1})$ is determined by assuming that B occurs at every location in the initial substring of length β of a copy of X_{i+1} and k times on each of the α boundaries. A lower bound is attained by assuming B never occurs in these positions, so

$$(3.2) \quad (1 - \epsilon_{i+1})\nu_{i+1}(B)\alpha|X_{i+1}| \leq N_m(B, l_{i+1}X_{i+1}) \leq (1 + \epsilon_{i+1})\nu_{i+1}(B)\alpha|X_{i+1}| + \beta + k\alpha.$$

We consider the case where B never occurs in any of the blocks X_j or on the borders for $j < i$. By applying (3.2), we arrive at

$$N_n(B, X) \geq (1 - \epsilon_i)\nu_i(B)l_i|X_i| + (1 - \epsilon_{i+1})\nu_{i+1}(B)\alpha|X_{i+1}|.$$

Assume that B occurs at every position in each of the X_j for $j < i$ and k times on each of the boundaries. Applying (3.2) again, we see that

$$\begin{aligned} N_n(B, X) &\leq (l_1|X_1| + \dots + l_{i-1}|X_{i-1}|) + (1 + \epsilon_i)\nu_i(B)l_i|X_i| + (1 + \epsilon_{i+1})\nu_{i+1}(B)\alpha|X_{i+1}| + \beta + k(l_i + 1)\alpha \\ &= (L_{i-1} + k(l_i + 1) + (1 + \epsilon_i)\nu_i(B)l_i|X_i|) + ((1 + \epsilon_{i+1})\nu_{i+1}(B)|X_{i+1}| + k)\alpha + \beta = \kappa. \end{aligned}$$

□

We will use the following rational functions, defined on $\mathbb{R}_0^+ \times \mathbb{R}_0^+$, to estimate (3.1):

$$\begin{aligned} f_{i,B}(w, z) &= \frac{(\phi_{L_{i-1}}(B) + \epsilon_i\nu_i(B)l_i|X_i|) + (\epsilon_{i+1}\nu_{i+1}(B)|X_{i+1}|)w + \nu_{i+1}(B)z}{\phi_{L_i}(B) + (\nu_{i+1}(B)|X_{i+1}|)w + \nu_{i+1}(B)z}, \\ g_{i,B}(w, z) &= \frac{(L_{i-1} + \epsilon_i\nu_i(B)l_i|X_i| + k(l_i + 1)) + (\epsilon_{i+1}\nu_{i+1}(B)|X_{i+1}| + k)w + z}{\phi_{L_i}(B) + (\nu_{i+1}(B)|X_{i+1}|)w + \nu_{i+1}(B)z}. \end{aligned}$$

Lemma 3.2. Let $k \in R(W)$, $k \leq k_i$, and $B \in \mathcal{D}_{\nu_i, k}$. Then

$$(3.3) \quad \left| \frac{N_n(B, X)}{\phi_n(B)} - 1 \right| < g_{i,B}(\alpha, \beta).$$

Proof. Using our lower bound from Lemma 3.1 on $N_n(B, x)$, $\frac{N_n(B, X)}{\phi_n(B)} - 1 < 0$, we arrive at the upper bound

$$\begin{aligned} (3.4) \quad \left| \frac{N_n(B, X)}{\phi_n(B)} - 1 \right| &\leq 1 - \frac{(1 - \epsilon_i)\nu_i(B)l_i|X_i| + (1 - \epsilon_{i+1})\nu_{i+1}(B)\alpha|X_{i+1}|}{\phi_n(B)} \\ &= \frac{\phi_n(B) - ((1 - \epsilon_i)\nu_i(B)l_i|X_i| + (1 - \epsilon_{i+1})\nu_{i+1}(B)\alpha|X_{i+1}|)}{\phi_n(B)} = f_{i,B}(\alpha, \beta). \end{aligned}$$

Similarly to (3.4) and using our upper bound from Lemma 3.1 for $N_n(B, x)$, we can conclude

$$\begin{aligned} \left| \frac{N_n(B, X)}{\phi_n(B)} - 1 \right| &\leq -1 + \frac{\kappa}{\phi_n(B)} = \frac{\kappa - \phi_n(B)}{\phi_n(B)} \\ &= \frac{1}{\phi_n(B)} \left(\left(\sum_{j=1}^{i-1} (1 - \nu_j(B))l_j|X_j| + k(l_i + 1) + \epsilon_i\nu_i(B)l_i|X_i| \right) + (\epsilon_{i+1}\nu_{i+1}(B)|X_{i+1}| + k)\alpha + (1 - \nu_{i+1}(B))\beta \right) \\ &< \frac{(L_{i-1} + \epsilon_i\nu_i(B)l_i|X_i| + k(l_i + 1)) + (\epsilon_{i+1}\nu_{i+1}(B)|X_{i+1}| + k)\alpha + \beta}{\phi_{L_i}(B) + (\nu_{i+1}(B)|X_{i+1}|)\alpha + \nu_{i+1}(B)\beta} = g_{i,B}(\alpha, \beta). \end{aligned}$$

So,

$$\left| \frac{N_n(B, X)}{\phi_n(B)} - 1 \right| < \max(f_{i,B}(\alpha, \beta), g_{i,B}(\alpha, \beta)).$$

However, since the numerator of $g_{i,B}(\alpha, \beta)$ is clearly greater than the numerator of $f_{i,B}(\alpha, \beta)$ and their denominators are the same we conclude that $f_{i,B}(\alpha, \beta) < g_{i,B}(\alpha, \beta)$. Therefore,

$$\left| \frac{N_n(B, X)}{\phi_n(B)} - 1 \right| < g_{i,B}(\alpha, \beta).$$

□

We will want to find a good bound for $g_{i,B}(w, z)$ where (w, z) ranges over values in $\{0, 1, \dots, l_{i+1}\} \times \{0, 1, \dots, |X_{i+1}| - 1\}$. Put

$$\tau_{W,B,i} = \sup(0, \sup\{t : \nu_{i+1}(B) \geq \nu_t(B)\}).$$

Note that $\tau_{W,B,i} < \infty$ as $(\nu_i(B))_i$ is eventually non-increasing. Set

$$\eta_{W,B,i} = \max(0, L_{\tau_{W,B,i}} \nu_{i+1}(B) - \phi_{L_i}).$$

Thus,

$$(3.5) \quad \nu_{i+1}(B) L_{i-1} \leq \phi_{L_{i-1}}(B) + \eta_{W,B,i}.$$

Lemma 3.3. If $k \in R(W)$, $|X_i| > 4k + \frac{2\eta_{W,B,i}}{\nu_i(B)}$, $|X_{i+1}| > \frac{k}{\nu_{i+1}(B)(\epsilon_i - \epsilon_{i+1})}$, $\epsilon_i < 1/2$, $l_i > 0$, $B \in \mathcal{D}_{\nu_i,k}$, $\mu_{i+1}(B) \leq \mu_i(B)$, and

$$(3.6) \quad (w, z) \in \{0, 1, \dots, l_{i+1}\} \times \{0, 1, \dots, |X_{i+1}| - 1\},$$

then

$$(3.7) \quad g_{i,B}(w, z) < g_{i,B}(0, |X_{i+1}|) = \frac{(L_{i-1} + \epsilon_i \nu_i(B) l_i |X_i| + k(l_i + 1)) + |X_{i+1}|}{\phi_{L_i}(B) + \nu_{i+1}(B) |X_{i+1}|}.$$

Proof. We note that $g_{i,B}(w, z)$ is a rational function of w and z of the form

$$g_{i,B}(w, z) = \frac{C + Dw + Ez}{F + Gw + Hz}$$

where

$$\begin{aligned} C &= L_{i-1} + \epsilon_i \nu_i(B) l_i |X_i| + k(l_i + 1), & D &= \epsilon_{i+1} \nu_{i+1}(B) |X_{i+1}| + k, & E &= 1, \\ F &= \phi_{L_i}(B), & G &= \nu_{i+1}(B) |X_{i+1}|, \text{ and } & H &= \nu_{i+1}(B). \end{aligned}$$

We will show that if we fix z , then $g_{i,B}(w, z)$ is a decreasing function of w and if we fix w , then $g_{i,B}(w, z)$ is an increasing function of z . To see this, we compute the partial derivatives:

$$\begin{aligned} \frac{\partial g_{i,B}}{\partial w}(w, z) &= \frac{D(F + Gw + Hz) - G(C + Dw + Ez)}{(F + Gw + Hz)^2} = \frac{D(F + Hz) - G(C + Ez)}{(F + Gw + Hz)^2}; \\ \frac{\partial g_{i,B}}{\partial z}(w, z) &= \frac{E(F + Gw + Hz) - H(C + Dw + Ez)}{(F + Gw + Hz)^2} = \frac{E(F + Gw) - H(C + Dw)}{(F + Gw + Hz)^2}. \end{aligned}$$

Thus, the sign of $\frac{\partial g_{i,B}}{\partial w}(w, z)$ does not depend on w and the sign of $\frac{\partial g_{i,B}}{\partial z}(w, z)$ does not depend on z . We will first show that $g_{i,B}(w, z)$ is an increasing function of z by verifying that

$$(3.8) \quad E(F + Gw) > H(C + Dw).$$

Let

$$\phi_i^*(B) = \nu_{i+1}(B) L_{i-1} + \epsilon_i \nu_i(B) \nu_{i+1}(B) l_i |X_i| + \nu_{i+1}(B) k(l_i + 1).$$

Thus, (3.8) can be written as

$$(3.9) \quad \phi_{L_i}(B) + \left[\nu_{i+1}(B) |X_{i+1}| w \right] > \phi_i^*(B) + \left[\nu_{i+1}(B) (\epsilon_{i+1} \nu_{i+1}(B) |X_{i+1}| + k) w \right].$$

We will verify this inequality in two steps by showing

$$\begin{aligned} \phi_{L_i}(B) &> \phi_i^*(B) \quad \text{and} \\ \nu_{i+1}(B) |X_{i+1}| w &> \nu_{i+1}(B) (\epsilon_{i+1} \nu_{i+1}(B) |X_{i+1}| + k) w. \end{aligned}$$

In order to show that $\phi_{L_i}(B) > \phi_i^*(B)$, we first note that

$$\phi_{L_i}(B) = \phi_{L_{i-1}}(B) + \nu_i(B) l_i |X_i|.$$

But by (3.5), we only need to show that

$$(3.10) \quad \nu_i(B) l_i |X_i| > \nu_{i+1}(B) (\epsilon_i \nu_i(B) l_i |X_i| + k(l_i + 1)) + \eta_{W,B,i}.$$

However, by rearranging terms, (3.10) is equivalent to

$$(3.11) \quad |X_i| > \frac{l_i + 1}{l_i} \cdot \frac{\nu_{i+1}(B)}{\nu_i(B)} \cdot \frac{1}{1 - \nu_{i+1}(B)\epsilon_i} \cdot k + \frac{\eta_{W,B,i}}{l_i \nu_i(B)(1 - \nu_{i+1}(B)\epsilon_i)}.$$

Since $l_i > 0$, we know that $(l_i + 1)/l_i \leq 2$. Since $\epsilon_i < 1/2$, we know that $(1 - \nu_{i+1}(B)\epsilon_i)^{-1} < 2$. Additionally, $\nu_{i+1}(B) \leq \nu_i(B)$ implies $\frac{\nu_{i+1}(B)}{\nu_i(B)} \leq 1$. Therefore,

$$\frac{l_i + 1}{l_i} \cdot \frac{\nu_{i+1}(B)}{\nu_i(B)} \cdot \frac{1}{1 - \nu_{i+1}(B)\epsilon_i} \cdot k + \frac{\eta_{W,B,i}}{l_i \nu_i(B)(1 - \nu_{i+1}(B)\epsilon_i)} < 2 \cdot 1 \cdot 2 \cdot k + \frac{2\eta_{W,B,i}}{\nu_i(B)} = 4k + \frac{2\eta_{W,B,i}}{\nu_i(B)}.$$

But, $|X_i| > 4k + \frac{2\eta_{W,B,i}}{\nu_i(B)}$. So (3.11) is satisfied and thus $\phi_{L_i}(B) > \phi_i^*(B)$.

The last step to verifying (3.9) is to show that

$$\nu_{i+1}(B)|X_{i+1}|w \geq \nu_{i+1}(B)(\epsilon_{i+1}\nu_{i+1}(B)|X_{i+1}| + k)w.$$

However, this is equivalent to

$$(3.12) \quad |X_{i+1}|w \geq (\epsilon_{i+1}\nu_{i+1}(B)|X_{i+1}| + k)w.$$

Clearly, (3.12) is true if $w = 0$. If $w > 0$ we can rewrite (3.12) as

$$|X_{i+1}| \geq \frac{1}{1 - \nu_{i+1}(B)\epsilon_{i+1}} \cdot k.$$

Similar to (3.11), $(1 - \nu_{i+1}(B)\epsilon_{i+1})^{-1}k \leq 2k < |X_i| \leq |X_{i+1}|$. Thus (3.8) is satisfied and $g_{i,B}(w, z)$ is an increasing function of z .

It will be more difficult to show that $\frac{\partial g_{i,B}}{\partial w}(w, z) < 0$ in a similar manner so we proceed as follows: because the sign of $\frac{\partial g_{i,B}}{\partial w}(w, z)$ does not depend on w , we will know that $g_{i,B}(w, z)$ is decreasing in w if for each z

$$\lim_{w \rightarrow \infty} g_{i,B}(w, z) < g_{i,B}(0, z).$$

Since $g_{i,B}(w, z)$ is an increasing function of z , we know for all z that $g_{i,B}(0, 0) < g_{i,B}(0, z)$. Hence, it is enough to show that

$$\lim_{w \rightarrow \infty} g_{i,B}(w, z) < g_{i,B}(0, 0).$$

Since $\lim_{w \rightarrow \infty} g_{i,B}(w, z) = D/G$ and $g_{i,B}(0, 0) = C/F$, it is sufficient to show that $CG > DF$:

$$(3.13) \quad (L_{i-1} + \epsilon_i \nu_i(B)l_i|X_i| + k(l_i + 1))\nu_{i+1}(B)|X_{i+1}| > (\epsilon_{i+1}\nu_{i+1}(B)|X_{i+1}| + k)\phi_{L_i}(B) \\ = (\epsilon_{i+1}\nu_{i+1}(B)|X_{i+1}| + k)(\phi_{L_{i-1}}(B) + \nu_i(B)l_i|X_i|) \\ \Leftrightarrow L_{i-1}\nu_{i+1}(B)|X_{i+1}| + \epsilon_i \nu_i(B)\nu_{i+1}(B)l_i|X_i||X_{i+1}| + k\nu_{i+1}(B)(l_i + 1)|X_{i+1}| \\ > (\epsilon_{i+1}\nu_{i+1}(B)|X_{i+1}| + k)\phi_{L_{i-1}}(B) + (\epsilon_{i+1}\nu_{i+1}(B)|X_{i+1}| + k)\nu_i(B)l_i|X_i|.$$

We will verify (3.13) by showing that

$$(3.14) \quad L_{i-1}\nu_{i+1}(B)|X_{i+1}| > (\epsilon_{i+1}\nu_{i+1}(B)|X_{i+1}| + k)\phi_{L_{i-1}}(B) \text{ and} \\ \epsilon_i \nu_i(B)\nu_{i+1}(B)l_i|X_i||X_{i+1}| > (\epsilon_{i+1}\nu_{i+1}(B)|X_{i+1}| + k)\nu_i(B)l_i|X_i|.$$

Since $L_{i-1} > \phi_{L_{i-1}}(B)$, in order to prove the first inequality of (3.14), it is enough to show that

$$\nu_{i+1}(B)|X_{i+1}| > \epsilon_{i+1}\nu_{i+1}(B)|X_{i+1}| + k,$$

which is equivalent to

$$|X_{i+1}| > \frac{k}{\nu_{i+1}(B)(1 - \epsilon_{i+1})}.$$

But $\epsilon_i < 1/2$, so

$$\frac{k}{\nu_{i+1}(B)(1 - \epsilon_{i+1})} < \frac{k}{\nu_{i+1}(B)(\epsilon_i - \epsilon_{i+1})} < |X_{i+1}|.$$

To verify the second inequality of (3.14) we cancel the common term $\nu_i(B)l_i|X_i|$ on each side to get

$$\epsilon_i \nu_{i+1}(B)|X_{i+1}| > \epsilon_{i+1}\nu_{i+1}(B)|X_{i+1}| + k,$$

which is equivalent to

$$|X_{i+1}| > \frac{k}{\nu_{i+1}(B)(\epsilon_i - \epsilon_{i+1})},$$

which is given in the hypotheses.

So, we may conclude that $g_{i,B}(w, z)$ is a decreasing function of w and an increasing function of z . We can achieve an upper bound on $g_{i,B}(w, z)$ by setting $w = 0$ and $z = |X_{i+1}|$:

$$g_{i,B}(w, z) < g_{i,B}(0, |X_{i+1}|) = \frac{(L_{i-1} + \epsilon_i \nu_i(B) l_i |X_i| + k(l_i + 1)) + |X_{i+1}|}{\phi_{L_i}(B) + \nu_{i+1}(B) |X_{i+1}|}.$$

□

Set

$$\epsilon'_i = \frac{(L_{i-1} + \epsilon_i \nu_i(B) l_i |X_i| + k(l_i + 1)) + |X_{i+1}|}{\phi_{L_i}(B) + \nu_{i+1}(B) |X_{i+1}|}.$$

Thus, under the conditions of Lemma 3.2 and Lemma 3.3,

$$(3.15) \quad \left| \frac{N_n(B, X)}{\phi_n(B)} - 1 \right| < \epsilon'_i$$

Lemma 3.4. If $k \in R(W)$ then $\lim_{i \rightarrow \infty} \epsilon'_i = 0$.

Proof. We first note that

$$(3.16) \quad \frac{l_i + 1}{l_i |X_i|} \leq \frac{2l_i}{l_i |X_i|} = \frac{2}{|X_i|} \rightarrow 0$$

by (2.2). Since (l_i) and $(|X_i|)$ are non-decreasing sequences

$$\frac{\sum_{j=1}^{i-2} l_j |X_j|}{l_i |X_i|} < \frac{i l_{i-2} |X_{i-2}|}{l_i |X_i|} = \left(\frac{l_{i-2} |X_{i-2}|}{l_{i-1} |X_{i-1}|} \right) \cdot \left(i \frac{l_{i-1} |X_{i-1}|}{l_i |X_i|} \right).$$

Hence, by (2.3), $\frac{l_{i-2} |X_{i-2}|}{l_{i-1} |X_{i-1}|} \rightarrow 0$ and $i b_i^k \frac{l_{i-1} |X_{i-1}|}{l_i |X_i|} \rightarrow 0$, so

$$(3.17) \quad \lim_{i \rightarrow \infty} \frac{\sum_{j=1}^{i-2} l_j |X_j|}{l_i |X_i|} = 0.$$

Thus,

$$(3.18) \quad \begin{aligned} \epsilon'_i &= \frac{\sum_{j=1}^{i-1} l_j |X_j| + \epsilon_i \nu_i(B) l_i |X_i| + |X_{i+1}| + k(l_i + 1)}{\sum_{j=1}^{i-1} j^{-k} l_j |X_j| + \nu_i(B) l_i |X_i| + \nu_{i+1}(B) |X_{i+1}|} < \frac{\sum_{j=1}^{i-1} l_j |X_j| + \epsilon_i \nu_i(B) l_i |X_i| + |X_{i+1}| + k(l_i + 1)}{\nu_i(B) l_i |X_i|} \\ &= \frac{\sum_{j=1}^{i-2} l_j |X_j|}{\nu_i(B) l_i |X_i|} + \frac{l_{i-1} |X_{i-1}|}{\nu_i(B) l_i |X_i|} + \epsilon_i + \frac{|X_{i+1}|}{\nu_i(B) l_i |X_i|} + \frac{k(l_i + 1)}{\nu_i(B) l_i |X_i|}. \end{aligned}$$

However, each term of (3.18) converges to 0 by (2.3), (2.4), (3.16), and (3.17). □

4. PROOF OF MAIN THEOREM 2.1

Let $B \in \mathcal{D}_{\mu,k}$ for $k \in R(W)$. Since $\frac{1}{\epsilon_{i-1} - \epsilon_i} = o(|X_i|)$, there exists n large enough so that $|X_i|$ and $|X_{i+1}|$ satisfy the hypotheses of Lemma 3.3.

For large enough n , we can apply Lemma 3.2 and Lemma 3.3 to conclude that

$$(4.1) \quad \left| \frac{N_n(B, X)}{\phi_n(B)} - 1 \right| < \epsilon'_i.$$

However, $\lim_{n \rightarrow \infty} i = \infty$. So, by Lemma 3.4

$$(4.2) \quad \lim_{n \rightarrow \infty} \epsilon'_i = 0.$$

Thus, by (4.1) and (4.2)

$$\lim_{n \rightarrow \infty} \left| \frac{N_n(B, X)}{\phi_n(B)} - 1 \right| = 0.$$

So,

$$\lim_{n \rightarrow \infty} \frac{N_n(B, X)}{\phi_n(B)} = 1.$$

Thus,

$$\lim_{n \rightarrow \infty} \frac{N_n(B, X)}{n} = \mu(B).$$

Now, suppose that $B \notin \mathcal{D}_{\mu, k}$. Note that

$$\sum_{B' \in \mathcal{D}_{\mu, k}} \lim_{n \rightarrow \infty} \frac{N_n(B', X)}{n} = \sum_{B' \in \mathcal{D}_{\mu, k}} \mu(B') = 1,$$

so $\lim_{n \rightarrow \infty} \frac{N_n(B, X)}{n} = 0 = \mu(B)$ and X is μ -normal of order k .

5. THE CONSTRUCTION

Our construction consists of the concatenation of all possible blocks of a fixed length. Since for the β -expansion we have that certain blocks of digits are not allowed, we have to introduce a padding in order to separate two successive blocks whose concatenation is not admissible. For example, if we take the golden mean as base, two successive ones are forbidden in the expansion. However, concatenating 1001 and 1010, which are admissible as such, yields the word 10011010 which is not admissible.

Therefore let c be the size of the padding and let $\mathcal{P}_{b, \omega} = \{P_1, \dots, P_{b^\omega}\}$ be the set of all possible blocks of length ω having digits in base b . Furthermore let $\tilde{P}_i := 0^c P_i$ be the i -th block including a padding of c zeros. Finally let $m_k = \min\{\mu(B) : B \in \mathcal{D}_{\mu, k}\}$ for $k \geq 1$ and M be an arbitrary large constant such that $M \geq \frac{1}{m_\omega}$.

The central tool for our construction will be the weighted concatenation of the blocks \tilde{P}_i , i.e.,

$$P_{b, \omega, M, c} := [M\mu(P_1)] \tilde{P}_1 [M\mu(P_2)] \tilde{P}_2 \dots [M\mu(P_{b^\omega})] \tilde{P}_{b^\omega}.$$

In the following we want to estimate the relative number of occurrences of a block within $P_{b, \omega, M, c}$. Therefore we have to estimate on the one hand the length of $P_{b, \omega, M, c}$ and on the other hand the number of occurrences of a block within $P_{b, \omega, M, c}$. We start with the estimation of the length of $P_{b, \omega, M, c}$.

Then, on the one hand, we get as upper bound

$$|P_{b, \omega, M, c}| = \sum_{i=1}^{b^\omega} [M\mu(P_i)] (c + \omega) \leq M(c + \omega) \sum_{i=1}^{b^\omega} \mu(P_i) + (c + \omega)b^\omega = (c + \omega)(M + b^\omega).$$

On the other hand we obtain as lower bound

$$|P_{b, \omega, M, c}| = \sum_{i=1}^{b^\omega} [M\mu(P_i)] (c + \omega) \geq M(c + \omega) \sum_{i=1}^{b^\omega} \mu(P_i) = M(c + \omega).$$

Now we want to give upper and lower bounds for the number of occurrences of a block B of length k in $P_{b, \omega, M, c}$.

- **Lower bound.** For the lower bound we only count the possible occurrences within a P_i . Therefore we can write every such P_i as $C_1 B C_2$ with possible empty C_1 or C_2 . Since the

block B is fixed, we let C_1 and C_2 vary over all possible blocks. Thus

$$\begin{aligned}
N(B, P_{b,\omega,M,c}) &\geq \sum_{m=0}^{\omega-k} \sum_{|C_1|=m} \sum_{|C_2|=\omega-k-m} [M\mu(C_1BC_2)] \\
&\geq M \sum_{m=0}^{\omega-k} \sum_{|C_1|=m} \sum_{|C_2|=\omega-k-m} \mu(C_1BC_2) \\
&= M \sum_{m=0}^{\omega-k} \sum_{|C_1|=m} \sum_{|C_2|=\omega-k-m-1} \sum_{d=0}^{b-1} \mu(C_1BC_2d) \\
&= \dots = M \sum_{m=0}^{\omega-k} \sum_{|C_1|=m-1} \sum_{d=0}^{b-1} \mu(dC_1B) \\
&= \dots = M \sum_{m=0}^{\omega-k} \mu(B) = (\omega - k + 1)M\mu(B),
\end{aligned}$$

where we have used that $\sum_{d=0}^{b-1} \mu(dA) = \sum_{d=0}^{b-1} \mu(Ad) = \mu(A)$.

- **Upper bound.** In order to provide an upper bound we will consider those occurrences within a block and those between two similar blocks separately. By using means similar to the above estimate we get an upper bound for the number of blocks occurring within a block \tilde{P}_i . In particular, we get

$$\sum_{C_1, C_2} [M\mu(C_1BC_2)] \leq \sum_{C_1, C_2} (M\mu(C_1BC_2) + 1) = \dots = (\omega + c - k + 1) (M\mu(B) + b^{\omega-k}),$$

Now we estimate the number of occurrences between two blocks for the upper bound. To this end we will distinguish two different cases whether the occurrence is between two identical blocks or two different ones. If the occurrence is between two identical blocks, then we have something like $P_i 0^c P_i = C_1BC_2$ with $\omega + c - k + 1 \leq |C_1| \leq \omega + c - 1$. Thus similar to above we get that there are

$$\begin{aligned}
&\sum_{m=\omega+c-k+1}^{\omega+c-1} \sum_{|C_1|=m} \sum_{|C_2|=2\omega+c-k-m} [M\mu(C_1BC_2)] \\
&\leq M \sum_{m=\omega+c-k+1}^{\omega+c-1} \sum_{|C_1|=m} \sum_{|C_2|=2\omega+c-k-m-1} \sum_{d=0}^{b-1} \mu(C_1BC_2d) + \sum_{m=\omega+c-k+1}^{\omega+c-1} b^{2\omega+c-k} \\
&= \dots = M \sum_{m=\omega+c-k+1}^{\omega+c-1} \sum_{|C_1|=m-1} \sum_{d=0}^{b-1} \mu(dC_1B) + (k-1)b^{2\omega+c-k} \\
&= \dots = M \sum_{m=\omega+c-k+1}^{\omega+c-1} \mu(B) + (k-1)b^{2\omega+c-k} \\
&= (k-1) (M\mu(B) + b^{2\omega+c-k})
\end{aligned}$$

occurrences between two identical blocks. We trivially estimate the number of occurrences between two different blocks by their total amount, which is $(k-1)b^\omega$. Thus we get that

$$\begin{aligned}
N(B, P_{b,\omega,M,c}) &\leq \sum_{m=0}^{\omega-k} \sum_{|C_1|=m-1} [M\mu(C_1BC_2)] + \sum_{m=\omega-k+1}^{\omega-1} \sum_{|C_1|=m} \sum_{|C_2|=2\omega-k-m} [M\mu(C_1BC_2)] + (k-1)b^\omega \\
&\leq (\omega + c - k + 1) (M\mu(B) + b^{\omega-k}) + (k-1) (M\mu(B) + b^{2\omega+c-k}) + (k-1)b^\omega \\
&\leq (\omega + c) (M\mu(B) + b^\omega) + (k-1)b^{2\omega+c-k}.
\end{aligned}$$

Finally we want to show the (ε, k) -normality of $P_{b,\omega,M,c}$. Thus it suffices to show that for all blocks B of length $m \leq k$ we have

$$(1 - \varepsilon)\mu(B) \leq \frac{N(B, P_{b,\omega,M,c})}{|P_{b,\omega,M,c}|} \leq (1 + \varepsilon)\mu(B)$$

Using our lower bound for the number of occurrences together with our upper bound for length we get that

$$\frac{N(B, P_{b,\omega,M,c})}{|P_{b,\omega,M,c}|} \geq \frac{(\omega - k + 1)M\mu(B)}{(\omega + c)(M + b^\omega)} \geq \mu(B) \left(1 - \frac{c + k - 1}{\omega + c}\right) \left(1 - \frac{b^\omega}{M + b^\omega}\right)$$

which implies the lower bound for

$$\varepsilon \leq \frac{c + k - 1}{\omega + c} + \frac{b^\omega}{M + b^\omega}.$$

On the other side an application of the upper bound for the number of occurrences together with the lower bound for the length yields

$$\frac{N(B, P_{b,\omega,M,c})}{|P_{b,\omega,M,c}|} \leq \frac{(\omega + c)(M\mu(B) + b^\omega) + (k - 1)b^{2\omega+c-k}}{(\omega + c)M} = \mu(B) + \frac{b^\omega}{M} + \frac{(k - 1)b^{2\omega+c-k}}{(\omega + c)M}$$

Putting these together we get that $P_{b,\omega,M,c}$ is (ε, k) -normal for

$$k \leq \omega \quad \text{and} \quad \varepsilon \leq \max \left(\frac{c + k - 1}{\omega + c} + \frac{b^\omega}{M + b^\omega}, \frac{1}{m_k} \left(\frac{b^\omega}{M} + \frac{(k - 1)b^{2\omega+c-k}}{(\omega + c)M} \right) \right).$$

6. EXAMPLES OF APPLICATION

We will use the following lemma which follows immediately from Main Theorem 2.1 and the previous section:

Lemma 6.1. *Let $W = ((l_i, \epsilon_i, k_i, \mu_i))_{i=1}^\infty$ be a block friendly family and $(X_i)_{i=1}^\infty$ a (W, μ) -good sequence. Suppose that $q_i \geq 2$ and M_i are sequences of positive integers such that $M_i \geq (\min\{\mu(B) : B \in \mathcal{D}_{\mu,i}\})^{-1}$ and*

$$(6.1) \quad q_i^{2^i} = o(M_i)$$

If $X_i = P_{q_i,i,M_i,c}$ with a fixed padding of $c \geq 0$ zeros, then $X = l_1X_1l_2X_2\cdots$ is μ -normal.⁴

6.1. Decimal expansion. Here, we use the weightings

$$\nu_i(j) = \begin{cases} \frac{1}{b} & \text{if } 0 \leq j \leq b - 1 \\ 0 & \text{if } j \geq b \end{cases}.$$

For $B = b_1 \dots b_k$, define $\nu_i(B) = \prod_{j=1}^k \nu_i(b_j)$ and let $\mu = \nu_1$. Let $q_i = b$, $M_i = b^{2^i} \log i$, $l_i = i^{2^i}$, and put $X_i = P_{b,i,M_i,0}$, so $ib^{2^i} \log i \leq |X_i| \leq ib^{2^i} \log i + ib^i$. A short computation shows that (2.2), (2.3), (2.4), and (6.1) hold with $\epsilon_i = 1/\sqrt{i}$. Thus, by Lemma 6.1, the numbers whose digits of its b -ary expansion are formed by $l_1X_1l_2X_2\cdots$ is normal in base b .

6.2. Continued fraction expansion. For a block $B = b_1 \dots b_k$, let Δ_B be the set of all real numbers in $(0, 1)$ whose first k digits of its continued fraction expansion are equal to B . Put

$$\mu(B) = \frac{1}{\log 2} \int_{\Delta_B} \frac{dx}{1+x}.$$

If there is an index j such that $b_j > i$, then let $\nu_i(B) = 0$. Let $S = \{j : b_j = i\}$. For $i < 8$, set $\nu_i(B) = \mu(B)$. For $i \geq 8$, if $S = \emptyset$, then let $\nu_i(B) = \mu(B)$. If $S \neq \emptyset$, then let

$$\nu_i(B) = \sum_{B'} \mu(B'),$$

where the sum is over all blocks $B' = b'_1 \dots b'_k$ such that for each index j in S , $b'_j \geq i$.

⁴Here $X_i = P_{q_i,\omega_i,M_i,c}$ is (ϵ_i, k_i, μ_i) -normal where $k_i = \lfloor \sqrt{i} \rfloor$ and $\omega_i = i$.

Put $m_i = \min_{B \in \mathcal{D}_{\nu_i}, |B|=i} \nu_i(B)$. We wish to find a lower bound for m_i . If $B = b_1 \dots b_k$, then let

$$\frac{p_k}{q_k} = \frac{1}{b_1 + \frac{1}{b_2 + \dots + \frac{1}{b_k}}}.$$

It is well known that $\lambda(\Delta_B) = \frac{1}{q_k(q_k + q_{k-1})}$ and $\mu(B) > \frac{1}{2 \log 2} \lambda(\Delta_B)$.

Thus, we may find a lower bound for m_i by minimizing $\frac{1}{q_i(q_i + q_{i-1})}$ for blocks B in \mathcal{D}_{ν_i} . The minimum will occur for $B = ii \dots i$. It is known that $q_n = iq_{n-1} + q_{n-2}$ if we set $q_0 = 1$ and $q_1 = i$. Set

$$r_1 = \frac{i + \sqrt{i^2 + 4}}{2}, r_2 = \frac{i - \sqrt{i^2 + 4}}{2}.$$

Then

$$q_n = \frac{r_1^{n+1} - r_2^{n+1}}{\sqrt{i^2 + 4}}.$$

Thus,

$$\frac{1}{q_i(q_i + q_{i-1})} = \frac{i^2 + 4}{(r_1^{i+1} - r_2^{i+1})(r_1^i + r_2^i) - (r_2^{i+1} - r_1^i)} > \frac{\log 2}{i^{2i}} \text{ for } i \geq 8.$$

Thus, $m_i > \frac{1}{2 \log 2} \left(\frac{\log 2}{i^{2i}} \right) = \frac{1}{2} i^{-2i}$. Let $M_i = 2i^{2i} \log i$ and $X_i = P_{i+1, i, M_i, 0}$ and Set $l_i = 0$ for $i < 8$ and $l_i = \lfloor i^2 \log i \rfloor$. Then

$$\frac{l_{i-1}}{l_i} \frac{|X_{i-1}|}{|X_i|} i < \frac{2(i-1)^{2i-1} + i^{i-1}}{2i^{2i}} = \left(1 - \frac{1}{i}\right)^{2i} \frac{1}{i-1} + \frac{1}{2i^{i+1}} \rightarrow 0$$

and

$$\frac{|X_{i+1}|}{l_i |X_i|} \leq \frac{2(i+1)^{2i+3} + (i+2)^{i+1}}{i^2 \log i \cdot 2i^{2i+1}} = \left(1 + \frac{1}{i}\right)^{2i} \frac{(i+1)^3}{i^3 \log i} + o(i^{-i}) \rightarrow 0.$$

By Lemma 6.1, the number whose digits of its continued fraction expansion are formed by $l_1 X_1 l_2 X_2 \dots$ is normal with respect to the continued fraction expansion.

6.3. Lüroth series expansion. ⁵ Put

$$\nu_i(j) = \begin{cases} 0 & j = 0, 1 \\ \frac{1}{j(j-1)} & 2 \leq j \leq i+1 \\ \frac{1}{i+1} & j = i+2 \\ 0 & j > i+2 \end{cases}$$

and

$$\mu(j) = \begin{cases} 0 & i = 0, 1 \\ \frac{1}{j(j-1)} & j \geq 2 \end{cases}$$

For $B = b_1 \dots b_k$, define $\nu_i(B) = \prod_{j=1}^k \nu_i(b_j)$ and $\mu(B) = \prod_{j=1}^k \mu(b_j)$. Clearly, $\nu_i \rightarrow \mu$. Next, we let $q_i = i+2$, $M_i = \max(3!^2, i^{2i} \log i)$, $l_i = \lfloor i^2 \log i \rfloor$, and $X_i = P_{i+2, i, M_i, 0}$. Note that for all $i \geq 1$

$$M_i \geq (i+1)!^2 > (\min\{\mu(B) : B \in \mathcal{D}_{\nu_i, i}\})^{-1}.$$

Conditions (2.2), (2.3), (2.4), and (6.1) hold. Thus, by Lemma 6.1, the numbers whose digits of its Lüroth series expansion are formed by $l_1 X_1 l_2 X_2 \dots$ is normal with respect to the Lüroth series expansion.

⁵This example may be modified to construct normal numbers with respect to *Generalized Lüroth series expansions* (see [6] for a definition of these expansions.)

6.4. Unfair coin. Let $p \in (0, 1)$, $p \neq 1/2$. Here, we use the weighting

$$\nu_i(j) = \begin{cases} p & \text{if } j = 0 \\ 1 - p & \text{if } j = 1 \\ 0 & \text{if } j > 1 \end{cases}.$$

For $B = b_1 \dots b_k$, define $\nu_i(B) = \prod_{j=1}^k \nu_i(b_j)$ and let $\mu = \nu_1$. Let

$$M_i = \left(\frac{1}{\min(p, 1-p)} \right)^{2i},$$

$l_i = i^{2i}$, and put $X_i = P_{2,i,M_i,0}$. Then X_i is $(1/\sqrt{i}, \sqrt{i}, \nu_i)$ -normal. By Lemma 6.1, X is μ -normal.

6.5. β -expansions. Let $\beta > 1$. Then every number $x \in [0, 1)$ has a greedy β -expansion given by the greedy algorithm (cf. Rényi [13]): set $r_0 = x$, and for $j \geq 1$, let $d_j = \lfloor \beta r_{j-1} \rfloor$ and $r_j = \{\beta r_{j-1}\}$. Then

$$x = \sum_{j \geq 1} d_j \beta^j,$$

where the d_j are integer digits in the alphabet $A_\beta = \{0, 1, \dots, \lfloor \beta \rfloor - 1\}$. We denote by $d(x) = d_1 d_2 d_3 \dots$ the greedy β -expansion of x .

Let D_β denote the set of greedy β -expansions of numbers in $[0, 1)$. A finite (resp. infinite) word is called β -admissible if it is a factor of an element (resp. an element) of D_β . Not every number is β -admissible and the β -expansion of 1 plays a central role in the characterization of all admissible sequences. Let $d_\beta(1) = b_1 b_2 \dots$ be the greedy β -expansion of 1. Since the expansion might be finite we define the quasi-greedy expansion $d_\beta^*(1)$ by

$$d_\beta^*(1) = \begin{cases} (b_1 b_2 \dots b_{t-1} (b_t - 1))^\omega & \text{if } d_\beta(1) = b_1 b_2 \dots b_t \text{ is finite,} \\ d_\beta(1) & \text{otherwise.} \end{cases}$$

Then Parry [11] could show the following

Lemma 6.1. Let $\beta > 1$ be a real number, and let s be an infinite sequence of non-negative integers. The sequence s belongs to D_β if and only if for all $k \geq 0$

$$\sigma^k(s) < d_\beta^*(1),$$

where σ is the shift transformation.

According to this result we call a number β such that $d_\beta(1)$ is eventually periodic a Parry number. In the present example we assume that β is such a number.

Since not all expansions are β -admissible we have to guarantee that if we concatenate the expansions of two blocks then this will generate an admissible sequence. In particular, let β be the golden mean, i.e. $\beta = \frac{1+\sqrt{5}}{2}$. Then the expansion of 1 is equal to $d_\beta(1) = 11$. In our construction we may take the two blocks 1001 and 1010, which are both admissible. However, if we concatenate them, we get the word 10011010, which is not admissible. In order to prevent this, we pad a certain amount of zeroes between two blocks. For the case of the golden mean, one zero is sufficient, since then we would get as concatenation 1001 0 1010 which is an admissible word.

In the following we will on the one hand provide an estimate for the number of zeroes we have to pad in order to get an admissible sequence. On the other hand we have to show that the constructed sequence is a normal number.

For the padding size we denote by $d_\beta(1) = b_1 \dots b_t (b_{t+1} \dots b_{t+p})^\omega$ the β -expansion of 1. If 1 has a finite expansion then we set $p = 0$. We are looking for the longest possible sequence of zeroes occurring in the expansion of 1. As one easily checks, the longest occurs if $b_1 = \dots = b_{t+p-1} = 0$ and $b_{t+p} \neq 0$. Thus we set the padding size c to be

$$c = t + p.$$

We wish to minimize the length of a cylinder set defined by a block of length ω . Define

$$\phi_\beta(\omega) = \begin{cases} 1 & \text{if } 1 \leq \omega \leq t \\ r & \text{if } t + (r-2)p \leq \omega \leq t + (r-1)p \end{cases}.$$

Then the length of this interval is at least $\beta^{-(t+\phi_\beta(\omega)p)}$. We use the fact that $\mu_\beta(I) \geq (1 - 1/\beta)\lambda(I)$ and put

$$M_i = \max \left(\frac{\beta^{t+\phi_\beta(i)p}}{1 - \frac{1}{\beta}}, \lceil \beta \rceil^{2i} \log i \right).$$

Put $X_i = P_{\lceil \beta \rceil, i, M_i, c}$ and $q_i = \lceil \beta \rceil$. Note that $\lim_{i \rightarrow \infty} \frac{\phi(i)}{i/p} = 1$, so for large i

$$(i+c)\lceil \beta \rceil^{2i} \log i \leq |X_i| \leq (i+c)(\lceil \beta \rceil^{2i} \log i + \lceil \beta \rceil^i)$$

Thus, for large i

$$|X_i| \approx i \lceil \beta \rceil^{2i} \log i.$$

Put $l_i = i^{2i}$ and the computation follows the same lines as above.

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(M. G. Madritsch) DEPARTMENT OF ANALYSIS AND COMPUTATIONAL NUMBER THEORY
 GRAZ UNIVERSITY OF TECHNOLOGY
 8010 GRAZ, AUSTRIA
E-mail address: `madritsch@math.tugraz.at`

(B. Mance) DEPARTMENT OF MATHEMATICS
 THE OHIO STATE UNIVERSITY
 231 WEST 18TH AVENUE
 COLUMBUS, OH 43210-1174, USA
E-mail address: `mance.8@osu.edu`